ANALYSIS of EUCLIDEAN ALGORITHMS

An Arithmetical Instance of Dynamical Analysis

Dynamical Analysis:=

Analysis of Algorithms + Dynamical Systems

Brigitte Vallée (CNRS and Université de Caen, France)

General Framework elaborated with the Caen Group : Ali Akhavi, Jérémie Bourdon, Julien Clément, Benoit Daireaux, Loick Lhote

outside Caen:
Viviane Baladi, Eda Cesaratto, Philippe Flajolet, Véronique Maume,

Dynamical Analysis –main principles.

Input.- A discrete algorithm.

Step 1.- Extend the discrete algorithm into a continuous process, i.e. a dynamical system. (X, V) X compact, $V : X \to X$, where the discrete alg. gives rise to particular trajectories.

Step 2.- Study this (continuous) Dynamical system, via its generic trajectories. A main tool: the transfer operator.

Step 3.- Coming back to the algorithm: Use the transfer operator as a generating operator, and prove that the particular trajectories due to the algorithm behave as the generic trajectories.

Output.- Probabilistic analysis of the Algorithm.

Plan of the talk.

I – Four types, six instances of Euclidean algorithms

II – The average-case analysis: The results.

III – The dynamical systems underlying the algorithms.

IV – The method: Dynamical Analysis

I – Four types, six instances of Euclidean algorithms

The Euclid Algorithm: the grand father of all the algorithms.

On the input (u, v), it computes the gcd of u and v, together with the Continued Fraction Expansion of u/v. $u_0 := v;\ u_1 := u; u_0 \geq u_1$

$$\left\{\begin{array}{lcccll} u_0 & = & m_1 u_1 & + & u_2 & 0 \leq u_2 < u_1 \\ u_1 & = & m_2 u_2 & + & u_3 & 0 \leq u_3 < u_2 \\ \ldots & = & \ldots & + & & \\ u_{p-2} & = & m_{p-1} u_{p-1} & + & u_p & 0 \leq u_p < u_{p-1} \\ u_{p-1} & = & m_p u_p & + & 0 & u_{p+1} = 0 \end{array}\right\}$$

u_p is the gcd of u and v, the m_i's are the digits. p is the depth.

CFE of $\frac{u}{v}$:
$$\frac{u}{v} = \cfrac{1}{m_1 + \cfrac{1}{m_2 + \cfrac{1}{\ddots + \cfrac{1}{m_p}}}},$$

The extended Euclid Algorithm

also returns the Bezout pair (r, s) for which $d = rv + su$.

It computes the sequence s_i defined by

$$s_0 = 0, \qquad s_1 = 1, \qquad s_i = s_{i-2} - s_{i-1} \cdot m_{i-1}, \quad 0 \leq i < p.$$

The last element s_p is the Bezout coefficient s.

Used for computing modular inverses: crucial in cryptography.

A Euclidean algorithm:=

any algorithm which performs a sequence of divisions $v = mu + r$.

Various possible types of Euclidean divisions

– MSB divisions [directed by the Most Significant Bits]

shorten integers on the left,

and provide a remainder r smaller than u,

(w.r.t the usual absolute value), i.e. with more zeroes on the left.

– LSB divisions [directed by the Least Significant Bits]

shorten integers on the right,

and provide a remainder r smaller than u

(w.r.t the dyadic absolute value), i.e. with more zeroes on the right.

– Mixed divisions

shorten integers both on the right and on the left,

with new zeroes both on the right and on the left.

Instances of MSB Algorithms.

– Variants according to the position of remainder r,

By Default: $v = mu + r$ with $0 \leq r < u$

By Excess: $v = mu - r$ with $0 \leq r < u$

Centered: $v = mu + \epsilon r$ with $\epsilon = \pm 1, \quad 0 \leq r \leq u/2$

– Subtractive Algorithm :

A division with quotient m can be replaced by m subtractions

While $v > u$ do $v := v - u$

An instance of a Mixed Algorithm.

The Subtractive Algorithm,

where the zeroes on the right are removed from the remainder

defines the Binary Algorithm.

Subtractive Gcd Algorithm.	Binary Gcd Algorithm.
Input. u, v; $v \geq u$	**Input.** u, v odd; $v \geq u$
While $(u \neq v)$ do	While $(u \neq v)$ do
While $v > u$ do	While $v > u$ do
	$k := \nu_2(v-u)$;
$v := v - u$	$v := \dfrac{v-u}{2^k}$;
Exchange u and v.	Exchange u and v.
Output. u (or v).	**Output.** u (or v).

The 2-adic valuation ν_2 counts the number of zeroes on the right

An instance of a LSB Algorithm.

On a pair (u, v) with v odd and u even,

with $\nu_2(u) = k$, of the form $u := 2^k\, u'$

the LSB division produces

– a quotient a odd, and $a \in [-(2^k - 1), 2^k - 1]$,

– and a remainder r with $\nu_2(r) > k$, of the form $r := 2^k\, r'$,

and writes $v = a \cdot u' + 2^k \cdot r'$.

The pair (r', u') satisfies

$$\nu_2(r') > \nu_2(u') = 0 \text{ and } \gcd(u, v) = \gcd(r', u').$$

It will be the new pair for the next step.

An execution of the LSB Algorithm on $(72001, 2011176)$

i	u_i [base 2]	a_i	k_i
1	11110101100000101000	-3	3
2	11001001101101010000	1	1
3	11000011000101000000	1	3
4	10011000111100000000	-1	1
5	11101001010100000000	-1	1
6	11000001001000000000	1	1
7	10001000110000000000	-1	1
8	10000010110000000000	1	1
9	11000000000000000	1	2
10	10000010000000000000	-1	1
11	10001000000000000000	1	1
12	11000000000000000000	-5	3
13	10000000000000000000	3	2

Comparison for five algorithms on the input $(2011176, 72001)$

Evolution of the remainders

Standard	Centered	By-Excess	Binary	LSB
67149	4852	4852	44849	51637
4852	779	779	1697	12485
4073	178	601	1697	2447
779	67	423	125	3733
178	23	245	125	1545
67	2	67	9	547
44	–	23	9	523
23	–	2	5	3
19	–	–	–	65
4	–	–	–	17
3	–		–	3

I – Four types, Six instances of Euclidean algorithms

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A general framework.

Each division–step of each algorithm uses a "digit" $d = (m, \epsilon, a, b)$, changes the old pair (u, v) into the new pair (r', u') as

$$u = 2^a \cdot u', \qquad v = m \cdot u' + \epsilon \cdot 2^b \cdot r'.$$

On integer pairs, it uses the matrix transformation $M_{[d]}$

$$\begin{pmatrix} u \\ v \end{pmatrix} = M_{[d]} \begin{pmatrix} r' \\ u' \end{pmatrix}, \qquad \text{with} \qquad M_{[d]} := \begin{pmatrix} 0 & 2^a \\ \epsilon\, 2^b & m \end{pmatrix}$$

and, on rationals (the old $x = u/v$ and the new $y = r'/u'$), it uses the LFT $h_{[d]}$,

$$x = h_{[d]}(y) \qquad \text{with} \quad h_{[d]}(y) = \frac{2^a}{m + \epsilon\, 2^b y}\ .$$

Then $|\det h_{[d]}| = 2^{a+b}$ involves the total number $a + b$ of binary shifts.

A generic execution.

On the input pair $(u, v) = (u_1, u_0)$, it is of the form

$$\left\{\begin{array}{lllllll} u_1 & := & 2^{-a_1}\, u_1, & u_0 & = & m_1\, u_1 & + & \epsilon_1\, 2^{b_1}\, u_2, \\ u_2 & := & 2^{-a_2}\, u_2, & u_1 & = & m_2\, u_2 & + & \epsilon_2\, 2^{b_2}\, u_3, \\ & & \ldots, & & & \ldots & & \\ u_i & := & 2^{-a_i}\, u_i, & u_{i-1} & = & m_i\, u_i & + & \epsilon_i\, 2^{b_i}\, u_{i+1} \\ & & \ldots, & & & \ldots & & \end{array}\right\},$$

and uses the sequence of digits $d_i := (m_i, \epsilon_i, a_i, b_i)$.

It stops at the p–th iteration with $u_{p+1} = \eta \cdot u_p$ $[\eta = 0$ or $\eta = 1]$.

Then $\gcd(u, v) = u_p$.

Cost of an execution.

Given a positive step–cost c defined on the set $\mathcal{D}$ of digits, consider the total cost C defined on the input (u, v) in an additive way as

$$C(u,v) := \sum_{i=1}^{p} c(d_i), \qquad\qquad d_i := (m_i, \epsilon_i, a_i, b_i)$$

The step–cost c is of moderate growth, when $c(d) = O(\log m)$

Main costs of moderate growth.

- if $c \equiv 1$, then $C = P$ is the number of iterations
- if c is the characteristic function $\mathbf{1}_{d_0}$ of a given digit d_0, then C is the number of occurrences of d_0 in the CF.
- if $c(d) = a + b$, then C is the total number of binary shifts.
- if $c(d) = \ell(m)$, the binary length of digit m, then C is the encoding length of the CF.

An important (non additive) cost.

The bit–complexity

$$B(u,v) := \sum_{i=1}^{p} \ell(m_i) \cdot \ell(u_i)$$

involves both digits d_i (via a cost of moderate length) and size of remainders u_i... The most precise cost

An Important Question.

Compare the behaviour of these various Euclidean algorithms with respect to different costs, and particularly the bit–complexity.

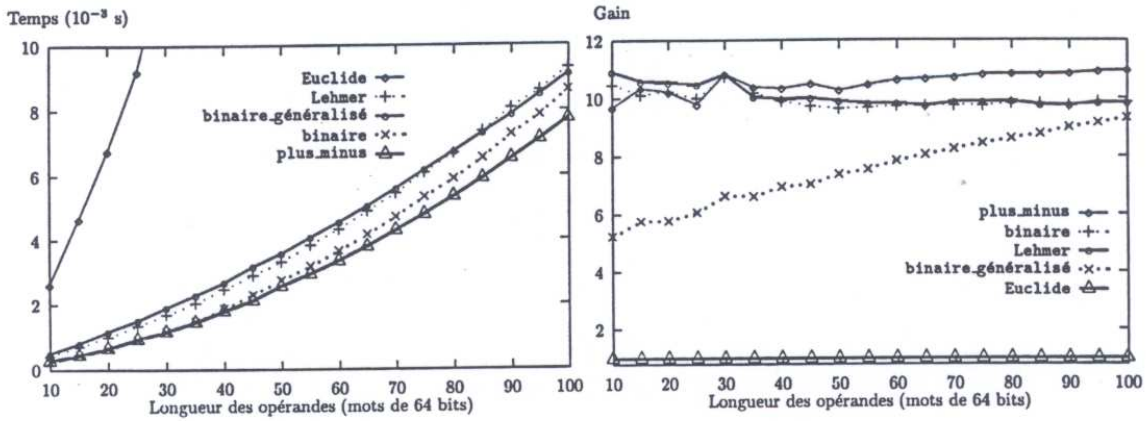

FIG. 9.20 – *Temps et gain par rapport à* `Euclide` *(station DEC).*

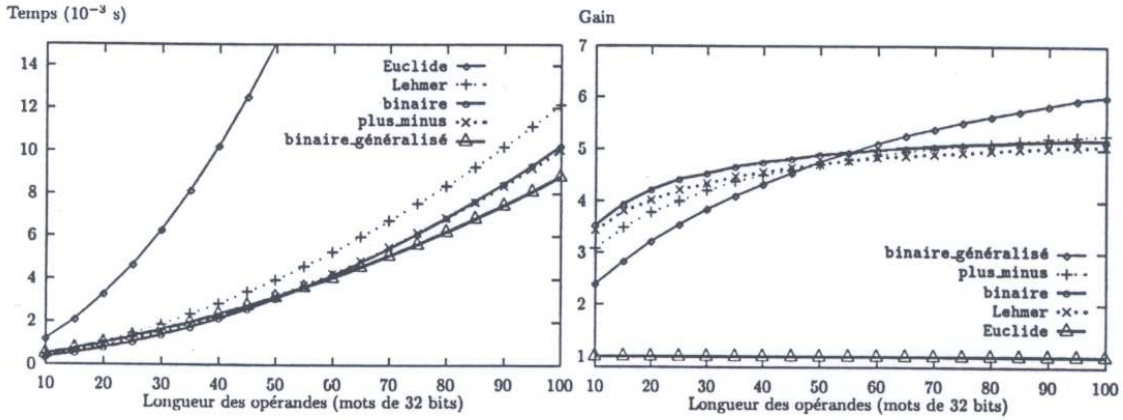

FIG. 9.21 – *Temps et gain par rapport à* `Euclide` *(station SUN).*

The analysis of the Euclidean Algorithms.

The length of an input (u, v) is $|(u, v)| \sim (u^2 + v^2)^{1/2}$.
Its size is $L(u, v) \sim (1/2) \lg(u^2 + v^2)$
If the set of all possible inputs (u, v) of the algorithm is Ω,
the algorithm is studied on $\Omega_n := \{(u, v) \in \Omega; \quad L(u, v) = n\}$
for $n \to \infty$.

Previous results, mostly in the average-case,
only for parameter P, and specific to particular algorithms...
well–described in Knuth's book (Tome II)

Heilbronn, Dixon, Rieger (70): Standard and Centered Alg.

Yao and Knuth (75): Subtractive Alg.

Brent (78): Binary Alg (partly heuristic),

Hensley (94) : A distributional study for the Standard Alg.

Stehlé and Zimmermann (05) : LSB Alg (experiments)

Then Dynamical Analysis [Caen group, 1995 $\rightarrow$?] provides

– a complete classification into two classes,

– the Fast Class ={Standard, Centered, Binary, LSB},

– the Slow Class = {By-Excess, Subtractive}.

– an average-case analysis of a broad class of costs,

– all the additive costs,

– and also the bit–complexity.

– a distributional analysis of a subclass of the Fast Class,

the Good Class = {Standard, Centered}.

Asymptotic gaussian laws hold for:

– P, and additive costs of moderate growth,

– the size of the remainders $\log u_i$ when $i \sim \delta P$,

– the bit-complexity of the extended Alg.

Here, focus on average-case results.

– For the Fast Class ={Standard, Centered, Binary, LSB } ,

– the mean values of costs C, B are linear wrt n,

– the mean bit-complexity is quadratic.

$$\mathbb{E}_n[P] \sim \frac{2\log 2}{h(\mathcal{S})} n, \qquad \mathbb{E}_n[C] \sim \frac{2\log 2}{h(\mathcal{S})} \mu[c]\, n, \qquad \mathbb{E}_n[B] \sim \frac{\log 2}{h(\mathcal{S})} \mu[\ell]\, n^2.$$

$h(\mathcal{S})$ is the entropy of the system, $\mu[c]$ the mean value of step–cost c.

– Moreover, these costs are concentrated: $\mathbb{E}_n[C^k] \sim E_n[C]^k$

– For the Slow Class = {By-Excess, Subtractive},

– the mean values of costs P, C are quadratic,

– the mean bit-complexity of B is cubic,

– the moments of order $k \geq 2$ are exponential: $\mathbb{E}_n[C^k] = \Theta(2^{n(k-1)})$.

The main constant $h(\mathcal{S})$ is the entropy of the Dynamical System.

A well-defined mathematical object, computable.

– Related to classical constants for the first two algs

$$h(\mathcal{S}) = \frac{\pi^2}{6\log 2} \sim 2.37 \text{ [Standard]}, \quad h(\mathcal{S}) = \frac{\pi^2}{6\log \phi} \sim 3.41 \text{ [Centered]}.$$

– For the LSB alg, $h(\mathcal{S}) = 4 - 2\gamma \sim 3.91$ involves the Lyapounov exponent γ of the set of random matrices, where

$$N_{a,k} = \frac{1}{2^k} \begin{pmatrix} 0 & 2^k \\ 2^k & a \end{pmatrix} \text{ with } k \geq 1, a \text{ odd}, |a| < 2^k \text{ is taken with prob. } 2^{-2k},$$

– For the Binary alg, $h(\mathcal{S}) = \pi^2 f(1) \sim 3.6$ involves the value $f(1)$ of the unique density which satisfies the functional equation

$$f(x) = \sum_{k\geq 1} \sum_{\substack{a \text{ odd} \\ 1\leq a<2^k}} \left(\frac{1}{2^k x + a}\right)^2 f\left(\frac{1}{2^k x + a}\right)$$

Precise comparisons between the four Fast Algorithms

Algs	Nb of iterations	Bit-complexity
Standard	$0.584\,n$	$1.242\,n^2$
Centered	$0.406\,n$	$1.126\,n^2$
(Ind.) Binary	$0.381\,n$	$0.720\,n^2$
LSB	$0.511\,n$	$1.115\,n^2$

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The Euclidean dynamical System (I).

The trace of the execution of the Euclid Algorithm on (u_1, u_0) is:

$$(u_1, u_0) \to (u_2, u_1) \to (u_3, u_2) \to \ldots \to (u_{p-1}, u_p) \to (u_{p+1}, u_p) = (0, u_p)$$

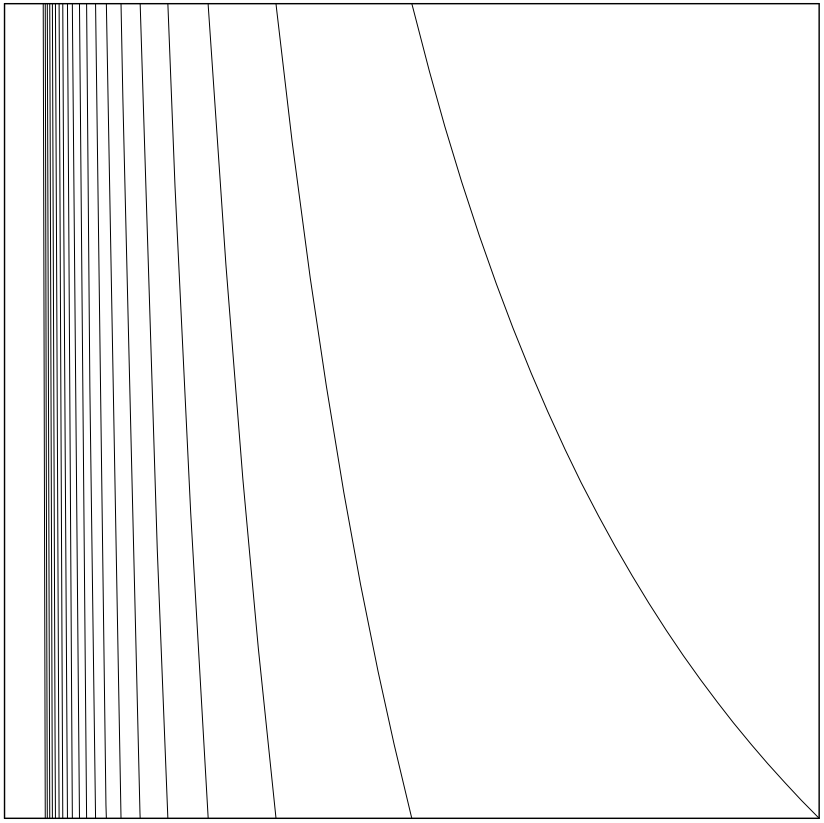
Replace the integer pair (u_i, u_{i-1}) by the rational $x_i := \dfrac{u_i}{u_{i-1}}$.

The division $u_{i-1} = m_i u_i + u_{i+1}$ is then written as

$$x_{i+1} = \frac{1}{x_i} - \left\lfloor \frac{1}{x_i} \right\rfloor \qquad \text{or} \qquad x_{i+1} = T(x_i), \qquad \text{where}$$

$$T : [0,1] \longrightarrow [0,1], \quad T(x) := \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor \quad \text{for} \quad x \neq 0, \quad T(0) = 0$$

An execution of the Euclidean Algorithm
= A rational trajectory of the Dynamical System $([0,1], T)$
= a trajectory that reaches 0.



The Euclidean dynamical System (II).

A dynamical system with a denumerable system of branches $(T_{[m]})_{m\geq 1}$,

$$T_{[m]} :]\frac{1}{m+1}, \frac{1}{m}[\longrightarrow]0,1[, \qquad T_{[m]}(x) := \frac{1}{x} - m$$

The set $\mathcal{H}$ of the inverse branches of T is

$$\mathcal{H} := \{\ h_{[m]} :]0,1[\longrightarrow]\frac{1}{m+1}, \frac{1}{m}[; \qquad h_{[m]}(x) := \frac{1}{m+x}\}$$

The set $\mathcal{H}$ builds one step of the CF's.

The set $\mathcal{H}^n$ is the set of the inverse branches of T^n;

it builds CF's of depth n.

The set $\mathcal{H}^\star := \bigcup \mathcal{H}^n$ builds all the (finite) CF's.

Dynamical Systems relative to MSB Algorithms.

A continuous dynamical system extends each discrete division:

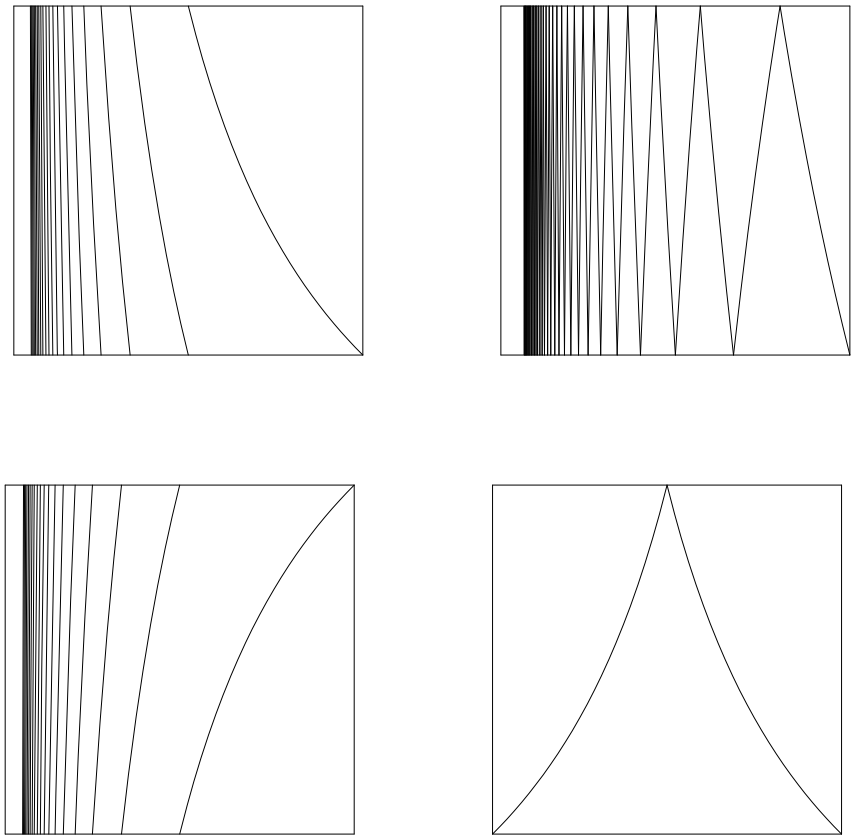
Replace the rational u/v by a generic real x [for the MSB Group]

Key Property : Expansiveness of branches

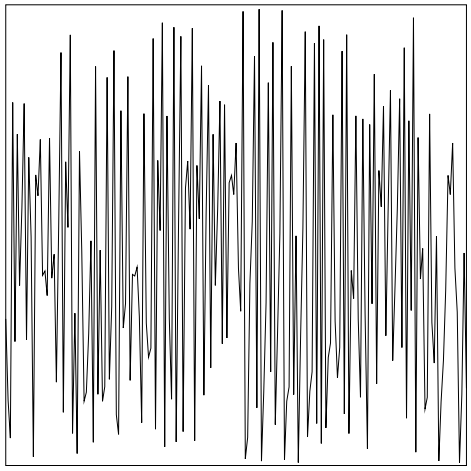
$$|T'(x)| \geq \rho > 1 \text{ for all } x \text{ in } \mathcal{I}$$

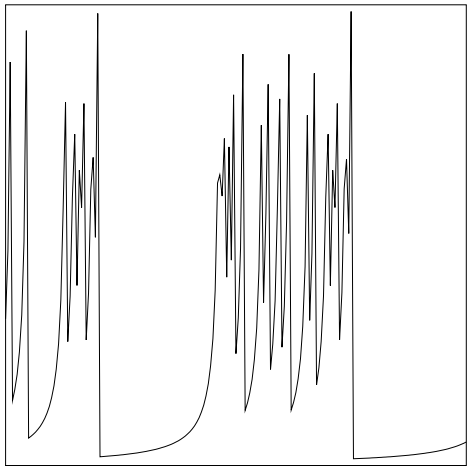
When true, this implies a chaotic behaviour for trajectories. The associated algorithms are Fast and belong to the Good Class

When this condition is violated at only one fixed point, this leads to intermittency phenomena. The associated algorithms are Slow.



Dynamical systems for MSB Algorithms; above, Standard and Centered; on the bottom, By-Excess and Subtractive. There exists an indifferent point for systems on the bottom line : $\eta = 1$ for (M) – $\eta = 0$ for (T).





Chaotic Orbit [Fast Class], Intermittent Orbit [SlowClass].

Induction Method.

For a DS with a "slow" branch relative to a slow interval J,
contract each part of the trajectory which belongs to J into one step.
This (often) permits to transform the slow DS into a fast one

While $x \in J$ do $x := T(x)$;
$S(x) := T(x)$;

The Induced DS of the Subtractive Alg = the DS of the Standard Alg.

The DS relative to the Binary Algorithm:

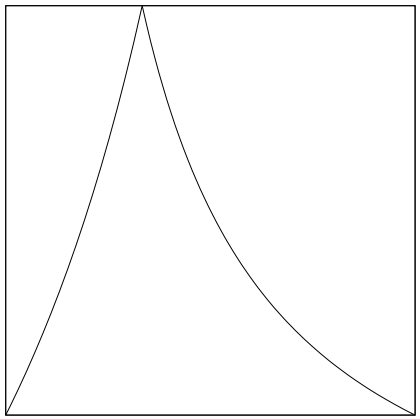
An instance of a probabilistic dynamic system.

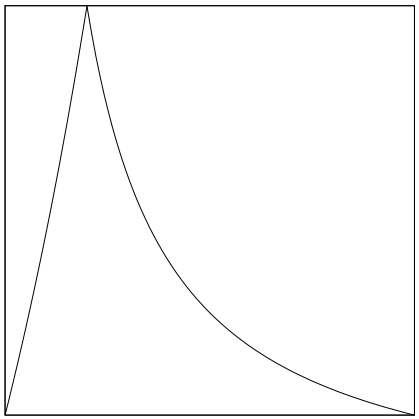
The 2–adic valuation ν (only defined on rationals) is extended to a real random variable ν with

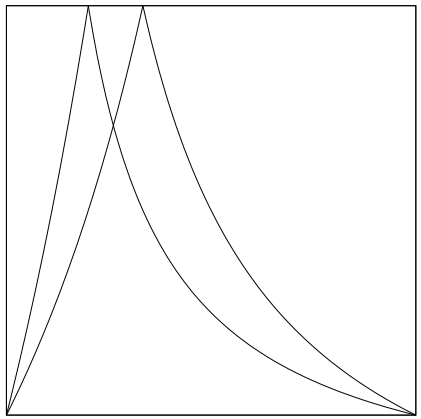
$$\mathbb{P}[\nu = k] = 1/2^k \qquad \text{for} \quad k \geq 1.$$

The Binary Dynamical system

for $k = 1$, $k = 2$; then with $k = 1$ and $k = 2$ on the same figure.







The induced DS of the Binary Algorithm:

A dynamical system where the set of inverse branches is

$$\mathcal{H} := \{h_{a,k} : x \mapsto \frac{1}{2^k x + a}, \qquad k \geq 1,\ a \text{ odd}, a < 2^k\}.$$

For a real $x \in]2^{-k_0}, 2^{-(k_0-1)}]$, the set of possible branches is

$$\{h_{a,k};\quad k \geq k_0,\ a \text{ odd}, a < 2^{k_0}\},$$

where each $h_{a,k}$ is chosen with probability 2^{-k}.

The probabilistic choice depends on the position of real x.

The LSB Dynamical System.

The remainders are not decreasing,

so that the rationals $x = u/v$ may belong to the whole $\mathbb{R}$.

Using the conjugaison with the tangent map leads to work inside the compact interval $J = [-\pi/2, +\pi/2]$ and to deal with inverse branches

$$h_{a,k} : x \mapsto \arctan\left(\frac{2^k}{2^k \tan x + a}\right)$$

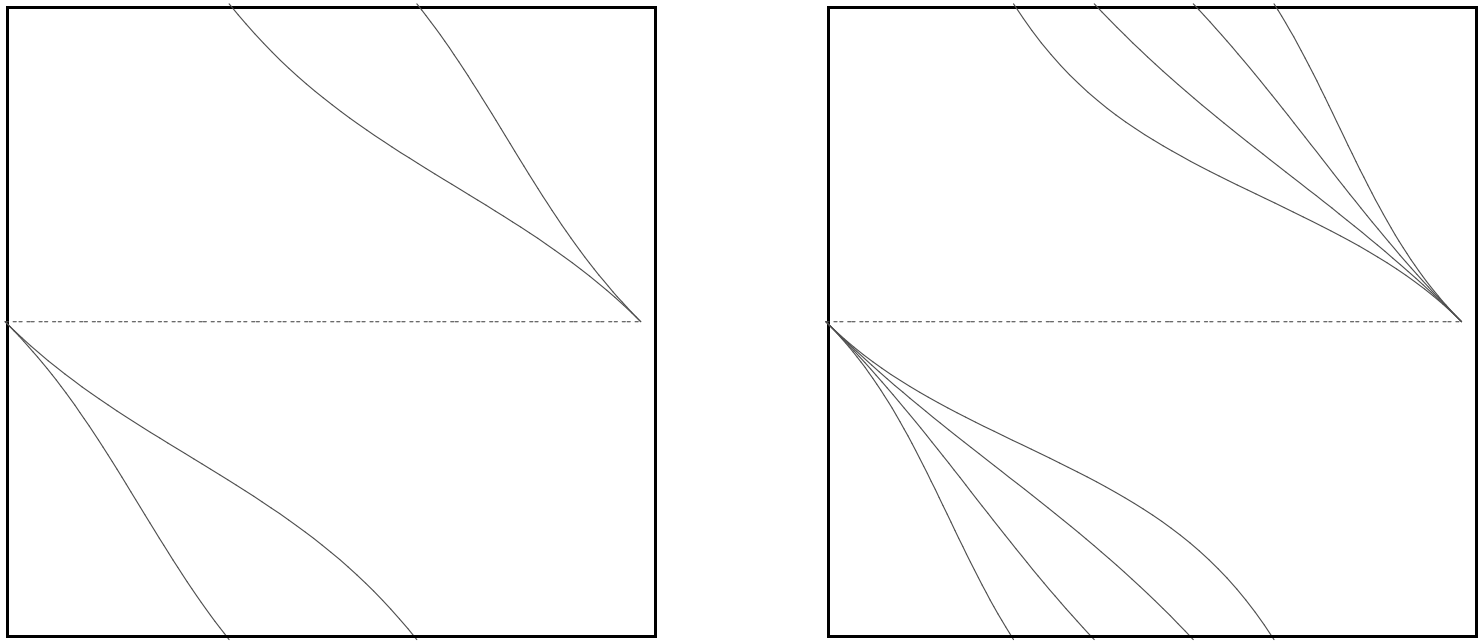
At each step of the process, each branch

$$h_{a,k}, \quad \text{with} \quad k \geq 1, \quad a \ \text{ odd}, \quad |a| < 2^k$$

is chosen with probability $1/2^{2k}$.

The probabilistic choice does not depend on the position of x.

This defines a system of iterated functions.



The DS relative to the LSB Alg.
On the left, for $k = 1[a = \pm 1]$ – On the right, for $k = 2[a = \pm 1, \pm 3]$.

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General principles of Dynamical Analysis.

Two objects:

The (discrete) Algorithm, the (continuous) Dynamical System

Two tools:

The generating function, The transfer operator

And their relations:

Geometric properties of the Dynamical System

$\Downarrow$

Spectral properties for the Transfer Operator

in a convenient functional space.

$\Downarrow$

Analytical properties of the Dirichlet series

$\Downarrow$

Asymptotic Analysis of the Algorithm

The density transformer $\mathbf{H}$ expresses the new density f_1 as a function of the old density f_0, as $f_1 = \mathbf{H}[f_0]$. It involves the set $\mathcal{H}$

$$\mathbf{H}[f](x) := \sum_{h \in \mathcal{H}} \delta_h \cdot |h'(x)| \cdot f \circ h(x) \qquad [\text{here, } \delta_h = \mathbb{P}[h]$$

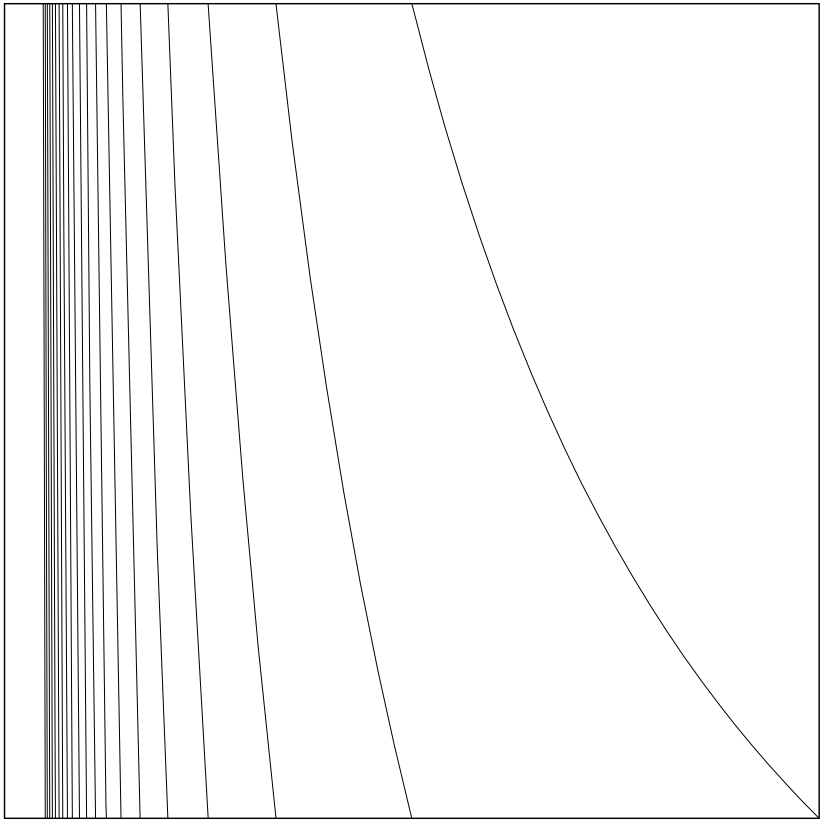
With a cost $c : \mathcal{H} \to \mathbf{R}^+$ defined by $c(h_{[d]}) := c(d)$ and extended to $\mathcal{H}^\star$ by additivity, it gives rise to the weighted transfer operator

$$\mathbf{H}_{s,w}[f](x) := \sum_{h \in \mathcal{H}} \delta_h^s \cdot \exp[wc(h)] \cdot |h'(x)|^s \cdot f \circ h(x)$$

$$\left\{ \begin{array}{c} \text{Multiplicative properties of the derivatives and probabilities} \\ \text{Additive properties of the cost} \end{array} \right\} \Longrightarrow$$

$$\mathbf{H}^k_{s,w}[f](x) := \sum_{h \in \mathcal{H}^k} \delta_h^s \cdot \exp[wc(h)] \cdot |h'(x)|^s \cdot f \circ h(x)$$

$\mathbf{H}^k_{s,w}$ generates all the weighted trajectories truncated at length k. $(I - \mathbf{H}_{s,w})^{-1} = \sum_{n \geq 0} \mathbf{H}^n_{s,w}$ generates all the finite trajectories



The Dirichlet series.

If Ω is the whole set of inputs, the Dirichlet generating function

$$S_C(s) = \sum_{(u,v)\in\Omega} \frac{C(u,v)}{|(u,v)|^{2s}} = \sum_{m\geq 1} \frac{c_m}{m^{2s}} \qquad \text{with } c_m := \sum_{\substack{(u,v)\in\Omega \\ |(u,v)|=m}} C(u,v)$$

is used for expressing the mean value $\mathbb{E}_n[C]$ of C on Ω_n, since

$$\mathbb{E}_n[C] = \frac{1}{|\Omega_n|} \sum_{m=2^{n-1}}^{2^n-1} c_m$$

The relation $S_C(s) = \dfrac{\partial}{\partial w} S_C(s,w)$,

$$\text{with} \qquad S_C(s,w) := \sum_{(u,v)\in\Omega} \frac{1}{|(u,v)|^{2s}} \exp[wC(u,v)],$$

proves that it is sufficient to deal with $S_C(s,w)$ and obtain an alternative expression for it.

Relation between the transfer operator and the Dirichlet series.

We need generating both the cost $C(u,v)$ and the length $|(u,v)|$. Any Euclid Algorithm provides a unique decomposition

$$u/v = h(\eta), \qquad \text{with} \qquad \eta = 0 \quad \text{or} \quad 1 \qquad \text{and} \qquad h \in \mathcal{H}^\star.$$

Then, $C(u,v) = c(h)$. There are two main choices for $|(u,v)|$,

$$|(u,v)| := \max(u,v) = v \qquad \text{or} \qquad |(u,v)|^2 = u^2 + v^2$$

according as the remainders are decreasing or not. In all the cases, (with the use of the tangent-conjugaison in the LSB case),

$$\frac{1}{|(u,v)|^{2s}} = \delta_h^s \cdot |h'(\eta)|^s,$$

Then, there is a nice relation between $S_C(s,w)$ and $\mathbf{H}_{s,w}$,

$$S_C(s,w) = (I - \mathbf{H}_{s,w})^{-1}[1](\eta),$$

and finally $$S_C(s) = (I - \mathbf{H}_s)^{-1} \circ \mathbf{H}_s^{[c]} \circ (I - \mathbf{H}_s)^{-1}[1](\eta)$$

Finally, an alternative expression of the Dirichlet series

$$S_C(s) := \sum_{(u,v)\in\Omega} \frac{C(u,v)}{|(u,v)|^{2s}} = (I - \mathbf{H}_s)^{-1} \circ \mathbf{H}_s^{[c]} \circ (I - \mathbf{H}_s)^{-1}[1](\eta)$$

as a function of two operators

$$\mathbf{H}_s^{[c]}[f](x) = \sum_{h\in\mathcal{H}} \delta_h^s \cdot c(h) \cdot |h'(x)|^s \cdot f \circ h(x)$$

and the quasi-inverse $(I - \mathbf{H}_s)^{-1}$ of the transfer operator $\mathbf{H}_s$,

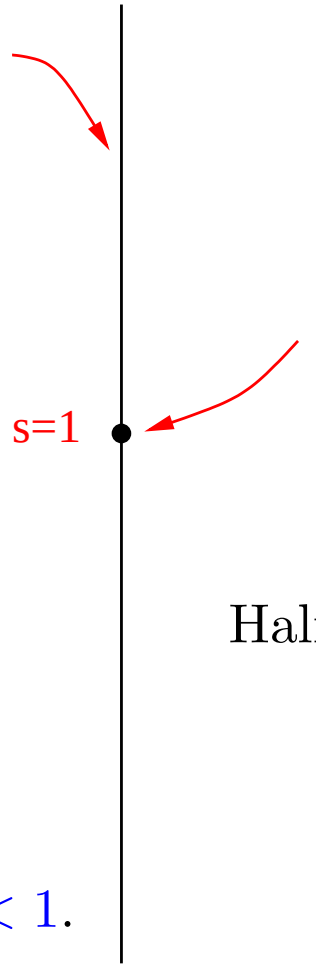
$$\mathbf{H}_s[f](x) := \sum_{h\in\mathcal{H}} \delta_h^s \cdot |h'(x)|^s \cdot f \circ h(x).$$

The quasi-inverse plays the leading rôle (except for Slow Alg): For extracting coefficients via Tauberian Theorems, we need to know the position and the nature of the (dominant) singularity of the quasi–inverse $(I - \mathbf{H}_s)^{-1}$.

Analytical properties of $(I - \mathbf{H}_s)^{-1}$ for applying Tauberian Theorems.

On the line $\Re s = 1$,

$s \neq 1$, $\mathbf{H}_s$ is analytic.



$s = 1$ is a (simple) pole

$$(I - \mathbf{H}_s)^{-1} \sim \frac{a}{s-1}$$

Half–plane of convergence $\Re s > 1$

No hypothesis needed

on the half–plane $\Re s < 1$.

Since $\mathbf{H}_1$ is the density transformer, $s = 1$ is the dominant singularity.

Expected spectral properties of $\mathbf{H}_s$

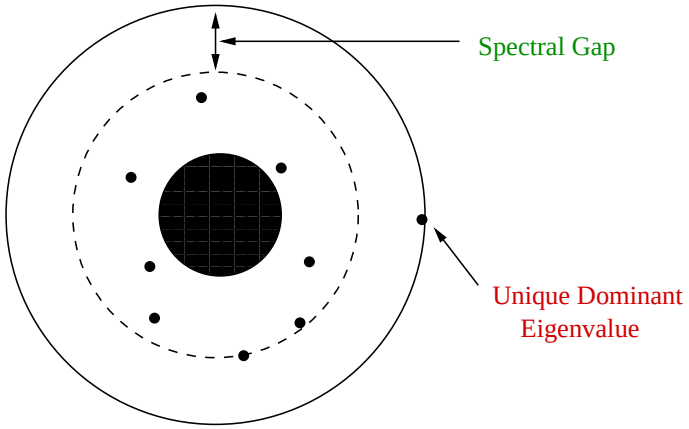
(i) *UDE* and *SG* for s near 1:

UDE – Unique dominant eigenvalue

SG – Existence of a spectral gap

(ii) Aperiodicity: for $\Re s = 1, s \neq 1$,

the spectral radius of $\mathbf{H}_s$ is < 1



On which functional space?

The answer depends on the DS,

and thus on the algorithm....

The functional space where the triple $UDE + SG +$ Aperiodicity holds.

Algs	Geometry of branches	Convenient Functional space
Good Class (Standard, Centered)	Contracting	$\mathcal{C}^1(\mathcal{I})$
Binary	Not contracting	The Hardy space $\mathcal{H}(\mathcal{D})$
LSB	Contracting on average	Various spaces: $\mathcal{C}^0(J), \mathcal{C}^1(J)$ Hölder $\mathbb{H}_\alpha(J)$
Slow Class (Subtractive, By-Excess)	An indifferent point	Induction $+ \mathcal{C}^1(\mathcal{I})$

The aperiodicity holds since the branches have not all the same form.

Conclusion.

Here, the average-case of a class of Euclid Algorithms....
which uses the transfer operator $\mathbf{H}_s$ of the underlying DS, with $\Re s \geq 1$.

Possible extensions.

Talks of Loick Lhote and Antonio Vera

– Distributional analysis of the Euclid algorithms
or Analysis of Fast variants of the Euclid Algorithms [Loick]
Use the same transfer operator $\mathbf{H}_s$, with its behaviour for $\Re s < 1$

– Study of the Gauss algorithm (for reducing lattices) [Antonio]
Use of an extension of the transfer operator $\mathbf{H}_s$,
which operates on functions of two variables, for $s \sim 2$