

**EXHAUSTIVE GENERATION OF SOME CLASSES OF PATTERN AVOIDING  
PERMUTATIONS USING SUCCESSION FUNCTIONS  
- EXTENDED ABSTRACT -**

PHAN-THUAN DO AND VINCENT VAJNOVSZKI

LE2I UMR-CNRS 5158, UNIVERSITÉ DE BOURGOGNE  
B.P. 47 870, 21078 DIJON-CEDEX FRANCE  
E-MAIL: {PHAN-THUAN.DO}{VVAJNOV}@U-BOURGOGNE.FR

ABSTRACT. ECO-method and its corresponding succession rules allow to recursively define and construct combinatorial objects. We introduce here *succession functions* which refine succession rules. They permit to construct general efficient algorithms for classes of combinatorial objects and to find some new classes more easily than previous methods. We apply succession function technique for a lot of classes of pattern avoiding permutations and derive general constant amortized time generating algorithms for them.

This paper is dedicated to Don Knuth who initiated 35 years ago the field of *pattern avoiding permutations*.

**Keywords** : Succession rule and function, pattern avoiding permutations, exhaustive generation, generating algorithm.

## 1. INTRODUCTION

The field of pattern avoiding permutations shows an increasing interest in the last two decades. A very powerful way to define and describe such class of permutations is the ECO method using succession rule and generating tree [1, 2, 3, 4, 5]. In this note, we refine this method; this allows us to define other classes, and to design very general and efficient algorithms for the exhaustive generation of these classes.

We refer the reader, for example to [2] and the reference therein, for the basic definitions concerning pattern avoiding permutations.

Occasionally a generating tree can be encoded by one or more classes of pattern avoiding permutations. In this case we can say that the succession rule of this tree corresponds to each of these classes. In this paper, we develop an other technique, *succession function*, which refine succession rule. Based on it, we derive two very general exhaustive generating algorithms for these classes. Sometimes this view helps to find new classes more easily than traditional methods. Here we bring together some basic results from [6], give many other succession functions and generalized succession functions corresponding to known and new classes. Our main original results are summarized in the four tables at the end of this note.

## 2. SUCCESSION FUNCTION

We begin by explaining the generating tree technique in the context of pattern avoidance. Let  $\mathfrak{S}_n$  be the set of permutations on  $[n] = \{1, 2, \dots, n\}$ . We represent a permutation  $\pi \in \mathfrak{S}_n$  in one line notation: i.e.  $\pi = \pi_1\pi_2 \dots \pi_n$  and denote by  $\mathfrak{S}_n(\tau)$  the set of permutations of  $\mathfrak{S}_n$  avoiding the

---

*Date*: October 17, 2007.

pattern  $\tau$ . The *sites* of  $\pi \in \mathfrak{S}_n$  are the positions between two consecutive entries, before the first and after the last entry; and they are numbered, from right to left, from 1 to  $n + 1$ . For a permutation  $\pi \in \mathfrak{S}_n(T)$ , where  $T$  is a set of forbidden patterns,  $i$  is an *active site* if the permutation obtained from  $\pi$  by inserting  $n + 1$  into its  $i$ th site is a permutation in  $\mathfrak{S}_{n+1}(T)$ ; we call such a permutation in  $\mathfrak{S}_{n+1}(T)$  a *son* of  $\pi$ . The active sites of a permutation  $\pi \in \mathfrak{S}_n(T)$  are *right justified* if the sites to the right of any active site are also active; and we denote by  $\chi_T(i, \pi)$  the number of active sites of the permutation obtained from  $\pi$  by inserting  $n + 1$  into its  $i$ th active site. Obviously, for any  $n > 1$  and  $\pi \in \mathfrak{S}_n(T)$ , by erasing  $n$  in  $\pi$  one obtains a permutation in  $\mathfrak{S}_{n-1}(T)$ ; or equivalently, any permutation in  $\mathfrak{S}_n(T)$  is obtained from a permutation in  $\mathfrak{S}_{n-1}(T)$  by inserting  $n$  in one of its active sites.

**2.1. Regular patterns.** A set of patterns  $T$  is called *regular* if

- the length one permutation  $1 \in \mathfrak{S}_1(T)$  has two sons (this condition can be dropped, it is used only to start the recursivity);
- all active sites are right justified;
- for any  $n \geq 1$  and  $\pi \in \mathfrak{S}_n(T)$ ,  $\chi_T(i, \pi)$  does not depend on  $\pi$  but solely on  $i$  and on the number  $k$  of active sites of  $\pi$ . In this case we denote  $\chi_T(i, \pi)$  by  $\chi_T(i, k)$  and we call it *succession function*.

Let  $T$  be a regular set of patterns characterized by its succession function  $\chi$ . The set of productions

$$(k) \rightsquigarrow (\chi_T(1, k))(\chi_T(2, k)) \dots (\chi_T(k, k));$$

or more concisely  $(k) \rightsquigarrow \cup_{i=1}^k (\chi_T(i, k))$ , for  $k \geq 1$ , is called the succession rule corresponding to the set of patterns  $T$ .

For any succession function (and generalized succession function later), one can derive a succession rule but the conversely is not true. So, we can say that the succession function gives more information than the succession rule for the class of combinatorial objects under consideration.

**Example 1.** The Catalan set of patterns  $T = \{312\}$  and  $T = \{321\}$  have the same succession rule  $(k) \rightsquigarrow (2)(3) \dots (k + 1)$  whereas they have different succession functions. See Table 1.

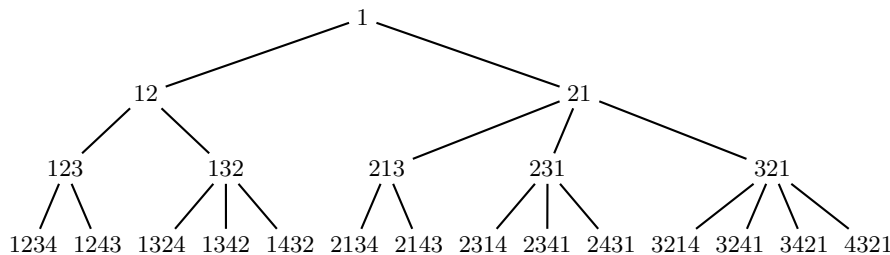


FIGURE 1. The first four levels of the generating tree corresponding to  $\chi(i, k) = i + 1$  producing  $S(312)$  permutations in 2.1.4. The generating tree coincides with the computation tree of the generating procedure.

**2.2.  $c$ -Regular patterns.** Often a succession function can not describe a generating tree and we need some additional information. Now we will associate to each permutation in  $\mathfrak{S}_n(T)$  an integer, called *color* and, we shall consider a no color permutation as a permutation with color 0. We extend

the previous  $\chi$  function in order to transform a triple  $(i, \pi, c)$  into a couple  $(\mu(i, \pi, c), \nu(i, \pi, c)) \in \mathbb{N}$  and we read it as:

if a permutation  $\pi \in \mathfrak{S}_n(T)$  has the color  $c$  then the insertion of  $n + 1$  in its  $i$ th active site produces a permutation  $\sigma \in \mathfrak{S}_{n+1}(T)$  with  $\mu(i, \pi, c)$  active sites and color  $\nu(i, \pi, c)$ .

A set of patterns  $T$  is called *colored regular* ( $c$ -regular for short) if

- the length one  $1 \in \mathfrak{S}_1(T)$  has two sons;
- all active sites are right justified;
- for any  $n \geq 1$  and  $\pi \in \mathfrak{S}_n(T)$ ,  $\chi_T(i, \pi, c)$  does not depend on  $\pi$  but only on  $i$ , on  $c$  and on the number  $k$  of active sites of  $\pi$ . In this case we denote  $\chi_T(i, \pi, c)$  by  $\chi_T(i, k, c)$  and  $\chi_T$  becomes a function  $\chi_T : \mathbb{N}^3 \rightarrow \mathbb{N}^2$ ; it generalizes the succession function  $\chi(i, k)$ .

Let  $T$  be a regular set of patterns characterized by its succession function  $\chi_T(i, k, c)$ . Let  $(\mu(i, k, c), \nu(i, k, c)) = \chi_T(i, k, c)$ . The set of productions

$$(k_c) \rightsquigarrow (\mu(1, k, c)_{\nu(1, k, c)}) (\mu(2, k, c)_{\nu(2, k, c)}) \dots (\mu(k, k, c)_{\nu(k, k, c)})$$

or more concisely

$$(k_c) \rightsquigarrow \cup_{i=1}^k (\mu(i, k, c)_{\nu(i, k, c)})$$

for  $k \geq 1$  is called *colored succession rule* which corresponds to the set of patterns  $T$ .

```

procedure Gen_Avoid(size, k)
local i
if size = n then
  Print( $\pi$ )
else
  size := size + 1
   $\pi$  := [ $\pi$ , size]
  gen_Avoid(size,  $\chi(1, k)$ )
  for i := 2 to k do
     $\pi$  :=  $\pi \cdot (size - i + 2, size - i + 1)$ 
    gen_Avoid(size,  $\chi(i, k)$ )
  end for
  for i := k downto 2 do
     $\pi$  :=  $\pi \cdot (size - i + 2, size - i + 1)$ 
  end for
end if
end procedure

```

(a)

```

procedure Gen_Avoid(size, k, c)
local i, u, v
if size = n then
  Print( $\pi$ )
else
  size := size + 1
   $\pi$  := [ $\pi$ , size]
  ( $\mu$ ,  $\nu$ ) :=  $\chi(1, k, c)$ 
  gen_Avoid(size,  $\mu$ ,  $\nu$ )
  for i := 2 to k do
     $\pi$  :=  $\pi \cdot (size - i + 2, size - i + 1)$ 
    (u, v) :=  $\chi(i, k, c)$ 
    gen_Avoid(size,  $\mu$ ,  $\nu$ )
  end for
  for i := k downto 2 do
     $\pi$  :=  $\pi \cdot (size - i + 2, size - i + 1)$ 
  end for
end if
end procedure

```

(b)

TABLE 1. (a) Algorithm for generating permutations avoiding a set of regular patterns characterized by the succession function  $\chi(i, k)$ . The first call is  $\text{Gen\_Avoid}(1, 2)$ ; (b) Algorithm for generating permutations avoiding a set of  $c$ -regular patterns characterized by the succession function  $\chi(i, k, c) = (\mu(i, k, c), \nu(i, k, c))$ . The first call is  $\text{Gen\_Avoid}(1, 2, 0)$ .  $\pi$  is initiated by the length 1 permutation. In particular, when only one color is allowed,  $\text{Gen\_Avoid}(size, k, c)$  becomes  $\text{Gen\_Avoid}(size, k)$ .

Let  $\sigma$  and  $\sigma'$  be the permutations in  $\mathfrak{S}_{n+1}(T)$  obtained from  $\pi \in \mathfrak{S}_n(T)$  by the insertion of  $n + 1$  into its  $i$ th and  $(i + 1)$ th active site. Then  $\sigma$  and  $\sigma'$  differ by a transposition, more precisely  $\sigma' = \sigma \cdot (n - i + 2, n - i + 1)$ . Generally, the insertion of an element in a list is not an efficient operation but the transposition of two element is an efficient elementary operation. Those are crucial in the efficaciousness of our generating algorithms. See our algorithms above.

## 3. LIST OF CLASSES TOGETHER WITH CORRESPONDING SUCCESSION FUNCTIONS

In the four tables below we give several classes of regular and  $c$ -regular patterns together with the  $\chi(i, k)$  and  $\chi(i, k, c)$  function. Seven classes not cited are new. Notice that  $i$  is always less or equal to  $k$ .  $T$  is a set of forbidden patterns.

Class	$T$	$\chi_T(i, k)$
$2^{n-1}$ [3]	{321, 312}	2
	{321, 231}	$k + 1$ if $i = 1$ 1 otherwise.
Pell numbers [3, 4]	{321, 3412, 4123}	3 if $i = 1$ 2 otherwise.
	{312, 4321, 3421}	3 if $i = 2$ 2 otherwise.
Even index Fibonacci numbers [3, 4]	{321, 3412}	$k + 1$ if $i = 1$ 2 otherwise.
	{321, 4123}	3 if $i = 1$ $i$ otherwise.
	{312, 4321}	3 if $k = 3$ and $i = 3$ $i + 1$ otherwise.
Catalan numbers [7]	{312}	$i + 1$
	{321}	$k + 1$ if $i = 1$ $i$ otherwise.
Schröder numbers [4]	{1234, 2134}	$k + 1$ if $i = 1$ or $i = 2$ $i$ otherwise.
	{1324, 2314}	$k + 1$ if $i = 1$ or $i = k$ $i + 1$ otherwise.
	{4123, 4213}	$k + 1$ if $i = k - 1$ or $i = k$ $i + 2$ otherwise.
Grand Dyck [4]	{1234, 1324, 2134, 2314}	$k + 1$ if $i = 1$ 3 if $i = 2$ $i$ otherwise.
	{1324, 2314, 3124, 3214}	3 if $i = 1$ $i + 1$ otherwise.
Motzkin numbers [4]	{321, 3 $\bar{1}$ 42}	$k + 1$ if $i = 1$ $i - 1$ otherwise.
The numbers of the left factors in Motzkin words [2]	{321, 4 $\bar{1}$ 523}	$k + 1$ if $i = 1$ 2 if $i = 2$ $i - 1$ otherwise.
Fibonacci numbers [8]	{321, 312, 231}	1 if $i = 2$ 2 otherwise.

TABLE 2. Several classes of regular patterns given by counting sequences together with the  $\chi(i, k)$  function.

## REFERENCES

- [1] F. R. K. Chung, R. L. Graham, V. E. Hoggatt, and M. Kleiman. The number of Baxter permutations. *J. Comb. Th., (A)* 24:382 – 394, 1978.
- [2] E. Barucci, A. Del Lungo, E. Pergola, and R. Pinzani. From Motzkin to Catalan permutations. *Discrete Mathematics*, 217, Issue 1-3:33 – 49, 2000.

Class	$T$	$\chi_T(i, k)$
A pattern of length 3 and a generalized pattern	$\{321, (p+1)12\dots p\}$ [3, 9]	$k+1$ if $i=1$ and $k < p$ $p$ if $i=1$ and $k=p$ $i$ otherwise.
	$\{321, p(p+1)12\dots(p-1)\}$ [3, 9]	$k+1$ if $i=1$ $i$ if $1 < i < p-1$ $p-1$ otherwise.
	$\{312, (p+1)p\dots 21\}$ [9]	$p$ if $k=p$ and $i=p$ $i+1$ otherwise.
A pattern of length 3, a pattern of length 4 and a generalized pattern	$\{321, 3412, (p+1)12\dots p\}$ [3]	$k+1$ if $i=1$ and $k < p$ $p$ if $i=1$ and $k=p$ $2$ otherwise.
Generalized Fibonacci numbers	$\{321, 231, (p+1)12\dots p\}$ [3]	$k+1$ if $i=1$ and $k < p$ $k$ if $i=1$ and $k=p$ $1$ otherwise.
A pattern of length 3 and two generalized pattern	$\{321, p(p+1)12\dots(p-1), (p+1)12\dots p\}$ [3]	$k+1$ if $i=1$ and $k < p$ $p$ if $i=1$ and $k=p$ $p-1$ if $i=p$ and $k=p$ $i$ otherwise.
	$\{321, p(p+1)12\dots(p-1), (m+1)12\dots m\}$ *	$k+1$ if $i=1$ and $k < m$ $m$ if $i=1$ and $k=m$ $i$ if $1 < i < p$ $p-1$ otherwise.
Generalized Schröder numbers	$\cup_{\tau \in \mathfrak{S}_{p-1}} \{(p+1)\tau p\}$ [10, 11, 12]	$k+1$ if $k \leq p$ or $i \geq k-p+1$ $i+p-1$ otherwise.
	$T = T_1 \cup T_2$ , where $T_1 = \{\sigma \in \mathfrak{S}_{p+1}   \sigma_1 = (p+1), \sigma_{p+1} = p\}$ , $T_2 = \{\sigma \in \mathfrak{S}_{p+1}   \sigma_1 = (p+1), \sigma_p = p\}$ *	$k+1$ if $k < p-1$ or $k-p+4 \leq i \leq k$ $p-1$ if $k \geq p-1$ and $1 \leq i \leq 2$ $i+p-3$ otherwise.
Generalized Motzkin numbers	$\{321, (p+2)\bar{1}(p+3)2\dots(p+1)\}$ [2]	$k+1$ if $i=1$ $i$ if $2 \leq i \leq p$ $i-1$ otherwise.

TABLE 3. Several classes of regular patterns given by variable length patterns together with the  $\chi(i, k)$  function. Notice that for each class of variable length pattern here, for some particular values of  $p$  or  $m$ , we retrieve some cases in the table 1 and 2. For example, the case  $T = \{321, 231, (p+1)12\dots p\}$ , if  $p=2$ , then we retrieve the case  $T = \{321, 231, 321\}$  in the table 1; and  $p = \infty$  corresponds to the case  $T = \{321, 231\}$  above.

- [3] E. Barucci, A. Bernini, and M. Poneti. From Fibonacci to Catalan permutations. *Pu.M.A.*, to appear, 2007.
- [4] O. Guibert. Combinatoire des permutations à motifs exclus en liaison avec mots, cartes planaires et tableaux de Young. *PhD Thesis, Université Bordeaux 1*, 1995.
- [5] L. Ferrari, E. Pergola, R. Pinzani, and S. Rinaldi. Jumping succession rules and their generating functions. *Discrete Math*, 271:29–50, 2003.
- [6] W.M.B. Dukes, M.F. Flanagan, T. Mansour, and V. Vajnovszki. Combinatorial Gray codes for classes of pattern avoiding permutations. *eprint arXiv:0704.2048*, 04/2007.
- [7] J. West. Generating trees and the Catalan and Schröder numbers. *Disc. Math.*, 146:247 – 262, 1994.
- [8] J.L. Baril and P.T. Do. Eco-generation for  $p$ -generalized Fibonacci and Lucas permutations. *Pu.M.A.*, to appear, 2007.
- [9] T. Chow and J. West. Forbidden sequences and Chebyshev polynomials. *Disc. Math.*, 204:119 – 128, 1999.
- [10] E. Barucci, A. Del Lungo, E. Pergola, and R. Pinzani. Permutations avoiding an increasing number of length-increasing forbidden subsequences. *Disc. Math. Theor. Comp. Sci.*, 4:31 – 44, 2000.

Class	$T$	$\mu_T(i, k, c)$	$\nu_T(i, k, c)$
A034943-Binomial transform of Padovan sequence	{312, 4321, 2431} *	3 if $i = k$ and $c \neq 1$ 2 otherwise.	1 if $i = 2$ and $(k = 3$ or $c = 1)$ 0 otherwise.
Even index Fibonacci numbers	{312, 2431} *	$i + 1$ if $i = 1$ or $(i = k$ and $c \neq 1)$ $i$ otherwise.	0 if $i = 1$ or $(i = k$ and $c \neq 1)$ 1 otherwise.
Grand Dyck	{2134, 2143, 2413, 4213} [4]	0 if $i < k - p$ $k + 1$ otherwise.	$k + 1 - i$ if $i \geq k - p$ 0 otherwise.
	{1234, 1243, 1423, 4123} [4]	0 if $i < k - p$ $k + 1$ otherwise.	$p + 1$ if $i = k$ $k - i$ if $k - p \leq i < k$ 0 otherwise.

TABLE 4. Several classes of  $c$ -regular patterns given by counting sequences together with the  $\chi(i, k, c) = (\mu(i, k, c), \nu(i, k, c))$  function.

Class	$T$	$\mu_T(i, k, c)$	$\nu_T(i, k, c)$
A pattern of length 3 and a generalized pattern	{312, $2(p + 1)p \dots 431$ } *	$k$ if $c = p - 3$ and $i = k$ $i + 1$ otherwise.	0 if $i = 1$ or $(c = 0$ and $i = k)$ $p - 3$ if $c = p - 3$ and $i = k$ 1 if $2 \leq i \leq k - c$ $1 + i - (k - c)$ otherwise.
Generalized Fibonacci numbers	{321, 312, $23 \dots (p + 1)1$ } [8]	1 if $c = p - 2$ and $i = 2$ 2 otherwise.	0 if $i = 1$ or $c = p - 2$ $c - 1$ otherwise.
A pattern of length 3, a pattern of length 4 and a generalized pattern	{321, 4123, $34 \dots (p + 1)12$ } [3]	2 if $i = 2$ or $(i = 3$ and $c = p - 3)$ 3 otherwise.	$c + 1$ if $i = 3$ and $c < p - 3$ 0 otherwise.
	{312, 4321, $34 \dots (p + 1)21$ } *	2 if $i = 1$ or $(i = 3$ and $c = p - 3)$ 3 otherwise.	$c + 1$ if $i = 3$ and $c < p - 3$ 0 otherwise.
	{312, 2431, $(p + 1)p \dots 21$ } *	$i + 1$ if $i = 1$ or $(i = k$ and $c = 0$ and $k < p)$ $i$ otherwise.	0 if $i = 1$ or $(i = k$ and $c = 0)$ 1 otherwise.

TABLE 5. Several classes of  $c$ -regular patterns given by variable length patterns together with the  $\chi(i, k, c) = (\mu(i, k, c), \nu(i, k, c))$  function. Notice that for each class of variable length pattern here, for some particular values of  $p$ , we retrieve some cases in previous tables.

- [11] D. Kremer. Permutations with forbidden subsequences and a generalized Schröder number. *Disc. Math.*, 218:121 – 130, 2000.
- [12] D. Kremer. Postscript: “Permutations with forbidden subsequences and a generalized Schröder number”. *Disc. Math.*, 270:332 – 333, 2003.

\* 7 new classes.